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A NOTE ON THE SHEAR CENTER PROBLEM FOR SHEAR-DEFORMABLE PLATES

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Abstract—It is shown that a minimum complementary energy analysis, in conjunction with Saint Venant type stress assumptions, for shear-deformable plates of variable thickness leads to a second order ordinary differential equation problem for the distribution of transverse shear. It is found that this equation is equi-dimensional for plates with linearly varying thickness. The ensuing exact solution implies an explicit expression for the location of the center of shear dependent on an appropriate dimensionless parameter involving cross-sectional dimensions and transverse twisting and shearing stiffness coefficients. Significant numerical consequences are encountered for plates which are relatively soft in transverse shear.

INTRODUCTION

Recent results for the shear center problem in the framework of Kirchhoff plate theory have left open the extent to which the effect of transverse shear-deformability becomes significant with increasing thickness–width ratio and with decreasing transverse shearing stiffness. In the following this problem is considered for an orthotropic plate, using the principle of minimum complementary energy in conjunction with Saint Venant type stress assumptions. In an earlier application of this approach to non-shear-deformable plates (Reissner, 1991), the numerical consequences were found to be quite close to corresponding results obtained by a more accurate and more complex analysis in which account was taken of anti-clastic curvature constraints by Reissner (1989) and Gu and Wan (1993). There is no reason to suppose that the same would not be true when transverse shear-deformability is taken into account.

The present approximate analysis reduces the problem to an ordinary second order differential equation. It is found that this equation can be solved explicitly for plates with linear widthwise–thickness variation, with a resultant closed-form expression for the shear center coordinate in terms of an appropriate dimensionless parameter.

FORMULATION

Consider a rectangular cantilever plate of span L and width a, with edges at $y = 0$, a and $x = 0$, L. The edge $x = 0$ is clamped and the edges $y = 0$, a are traction free. The edge $x = L$ is stipulated to deflect uniformly by an amount W, in conjunction with two conditions of absent bending moments and edgewise rotational displacements.

The minimum complementary energy formulation for a shear-deformable plate is, for this problem, given in terms of stress couples M_x, M_y, M_t and stress resultants Q_x, Q_y by the variational equation

$$\delta\left[\int_0^L\int_0^a V(M_x, M_y, M_t, Q_x, Q_y)\,\mathrm{d}x\,\mathrm{d}y - W\int_0^a (Q_x)_L\,\mathrm{d}y\right] = 0. \tag{1}$$

The complementary energy density V is for a linearly elastic orthotropic plate, to which attention is restricted in what follows, of the form

$$V = \frac{D_y M_x^2 + D_x M_y^2 + 2D_v M_x M_y}{2(D_x D_y - D_v^2)} + \frac{M_t^2}{2D_t} + \frac{B_x Q_x^2 + B_y Q_y^2}{2}, \tag{2}$$

where

$$D_y = D_x = \frac{Eh^3}{12(1-\nu^2)}, \quad D_v = \nu D_x, \quad D_t = (1-\nu)\, D_x, \quad B_x = B_y = \frac{6}{5Gh} \tag{3}$$

for an isotropic homogeneous plate of thickness $h = h(y)$.

Equation (1) is associated with constraint differential equations

$$Q_x = M_{x,x} + M_{t,y}, \quad Q_y = M_{t,x} + M_{y,y}, \quad Q_{x,x} + Q_{y,y} = 0 \tag{4}$$

and with constraint boundary conditions, which in this case are

$$y = 0, a; \quad Q_y = M_y = M_t = 0; \quad M_x(L, y) = 0. \tag{5}$$

Equations (1) and (5) will be used in conjunction with the Saint Venant type assumptions

$$M_y = 0, \quad Q_y = 0 \tag{6}$$

for an approximate determination of a force Q and a torque T,

$$Q = \int_0^a Q_x \, \mathrm{d}y, \quad T = \int_0^a (Q_x y - M_t)\, \mathrm{d}y \tag{7}$$

and a shear center coordinate

$$y_s = T/Q. \tag{8}$$

REDUCTION

The three relations in eqn (4), in conjunction with eqns (5) and (6), give the following as expressions for Q_x, M_t, and M_x:

$$Q_x = Q_x(y), \quad M_t = M_t(y), \quad M_x = (Q_x - M_t')(x - L) \tag{9}$$

with the prime indicating differentiation with respect to y.

The introduction of eqns (6) and (9) into eqns (2) and (1) leads to the one-dimensional variational equation

$$\delta \int_0^a \left[\frac{L^3}{3} \frac{(Q_x - M_t')^2}{2D_b} + \frac{LM_t^2}{2D_t} + \frac{LB_x Q_x^2}{2} - WQ_x \right] \mathrm{d}y = 0 \tag{10}$$

with constraint boundary conditions

$$M_t(0) = M_t(a) = 0 \tag{11}$$

and with $D_b = (1 - D_v^2/D_x D_y)\, D_x$.

The Euler differential equations of (10) are

$$\frac{Q_x - M_t'}{3D_b} + \frac{B_x Q_x}{L^2} = \frac{W}{L^3}, \quad \left(\frac{Q_x - M_t'}{3D_b}\right)' + \frac{M_t}{L^2 D_t} = 0. \tag{12}$$

While it would be possible to reduce eqn (12) to one second order equation for M_t, it is preferable to proceed as follows. Introduction of $Q_x - M_t'$ from the first relation in (12) into the second gives the following as an expression for M_t in terms of Q_x:

$$M_t = D_t(B_x Q_x)'. \tag{13}$$

With eqn (13) the first relation in eqn (12), together with eqn (11), leaves the boundary value problem

$$\left(1 + \frac{3B_x D_b}{L^2}\right) Q_x - [D_t(B_x Q_x)']' = \frac{3D_b}{L^3} W \tag{14}$$

$$y = 0, a; \quad D_t(B_x Q_x)' = 0. \tag{15}$$

It is allowable, for simplicity's sake, to set $3W/L^3 = 1$. Furthermore, except for terms of relative order h^2/L^2, eqn (14) may be replaced by

$$Q_x - [D_t(B_x Q_x)']' = D_b. \tag{14'}$$

With this and upon observation of eqns (15) and (13), the expressions for Q and T become

$$Q = \int_0^a D_b \, dy, \quad T = \int_0^a D_b y \, dy - 2\int_0^a D_t(B_x Q_x)' \, dy. \tag{16}$$

Upon setting $B_x = 0$, the non-shear-deformational plate theory result

$$y_s = \int_0^a D_b y \, dy \Big/ \int_0^a D_b \, dy$$

becomes an immediate consequence.

A CLOSED-FORM SOLUTION

It is possible to obtain an explicit solution in closed form for plates for which

$$D_b = D_{b0}\eta^3, \quad D_t = D_{t0}\eta^3, \quad B_x = B_{x0}\eta^{-1}, \quad \eta = y/a. \tag{17}$$

In view of eqn (3), this includes the case of a homogenous orthotropic plate of linearily varying thickness $h = h_0\eta$.

Upon setting

$$Q_x = D_{b0} f(\eta), \quad \varepsilon = \sqrt{D_{t0} B_{x0}}/a, \tag{18}$$

eqns (14) and (15) assume the equi-dimensional form

$$f - \varepsilon^2 [\eta^3 (\eta^{-1} f)^{\cdot}]^{\cdot} = \eta^3, \quad [\eta^3 (\eta^{-1} f)^{\cdot}]_{0,1} = 0, \tag{19}$$

with dots indicating differentiation with respect to η.

An inspection reveals that the differential equation in (19) is explicitly solvable in terms of suitable powers of η. Upon satisfaction of the two boundary conditions the solution comes out to be

$$f = \frac{1}{1-8\varepsilon^2}\left(\eta^3 - \frac{2\eta^p}{p-1}\right), \quad p = \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} \tag{20}$$

and

$$D_t(B_x Q_x)' = D_{b0}\,\varepsilon^2\eta^3(\eta^{-1}f)^{\cdot} = D_{b0}2\varepsilon^2\frac{\eta^4-\eta^{p+1}}{1-8\varepsilon^2}. \tag{21}$$

The introduction of eqns (17) and (21) into eqn (16) gives, after some transformations,

$$\frac{Q}{D_{b0}} = \frac{a^4}{4}, \quad \frac{T}{D_{b0}} = \frac{a^5}{5}\left(1 - \frac{4\varepsilon^2}{1+5\varepsilon\sqrt{1+\varepsilon^2}+7\varepsilon^2}\right) \tag{22}$$

and therewith, in accordance with eqn (8),

$$\frac{y_s}{a} = \frac{4}{5}\left(1 - \frac{4\varepsilon^2}{1+5\varepsilon\sqrt{1+\varepsilon^2}+7\varepsilon^2}\right). \tag{23}$$

An impression of the significance of the effect of transverse shear deformation may be gained by considering a homogenous orthotropic plate for which

$$D_{t0} = \frac{Gh_0^3}{6}, \quad B_{x0} = \frac{6}{5G_t h_0}, \quad \varepsilon^2 = \frac{1}{5}\frac{G}{G_t}\frac{h_0^2}{a^2}, \tag{24}$$

with $G = G_t = E/2(1+\nu)$ for the case of isotropy. As ε increases, the value of y_s/a decreases, at first approaching the cross-sectional centroid value $y_c/a = 2/3$ from above. For sufficiently large ε values, y_s/a becomes smaller than y_c/a. For example, when $\varepsilon = 1$ then $y_s/a = 0.59$. In this connection it is worth noting that for a "plate", for which the cross-section is an equilateral triangle with $h_0/a = 2/\sqrt{3}$ and for which a plate theoretical analysis is clearly not rational, eqn (23) gives $y_s/a = 0.647$ when $G/G_t = 1$ and $\varepsilon^2 = 4/15$, in place of the correct value $y_s/a = 2/3$.

It seems reasonable to limit the applicability of eqn (23) by the stipulation that $h_0/a \leqslant 1/2$. This does not preclude the possibility of significant numerical effects, for sufficiently large values of G/G_t.

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